April 9, 2018

MATH2050C Quiz 4a

1. (10 marks) Find all points of continuity for the function $f(x) = x, x \in \mathbb{Q}$ and $f(x) = 0$ elsewhere. You need to justify your result.

   Solution. The function f is continuous at $x = 0$ and discontinuous elsewhere.

   For $\varepsilon > 0$, we let $\delta = \varepsilon$, then trivially $|f(x) - f(0)| = |f(x)| = |x| < \varepsilon$ whenever $x \in \mathbb{Q}$ and $|x| < \delta$. On the other hand, $|f(x) - f(0)| = |0 - 0| = 0 < \varepsilon$ whenever $x \in \mathbb{R} \setminus \mathbb{Q}$ and $|x| < \delta$. We conclude f is continuous at $x = 0$.

   Next, when x_0 is a non-zero rational number, we pick a sequence of irrational numbers $x_n \to x_0$, then $f(x_n) = 0$ is a constant zero sequence which does not converge to $f(x_0) = x_0 \neq 0$. On the other, when x_0 is a non-zero irrational number, we pick a sequence of rational number $z_n \to x_0$, then $f(z_n) = z_n \to x_0$, which is not equal to $f(x_0) = 0$. Hence no matter the non-zero x_0 is rational or irrational, f is discontinuous at it.

2. (10 marks) Explain the continuity of the function

   $$g(x) = \sin\left(\frac{1+x^2}{\sqrt{x}-1}\right) , \quad x \in (1, \infty) .$$

   You should point out the facts you use in your explanation.

   Solution. We observe first that the functions $1 + x^2$ and $\sqrt{x} - 1$ are continuous on $(1, \infty)$ (no need to explain further). Next, as the quotient of two continuous functions, $h(x) = (1 + x^2)/(\sqrt{x} - 1)$ is continuous on $(1, \infty)$. (We introduce the notation h for later use.) Finally, as the composition of two continuous functions is continuous, $g(x) = \sin(h(x))$ is continuous on $(1, \infty)$. (The fact that the sine function is continuous everywhere is understood and need not point out).